

Manifolds with affine connection ∇

Two languages:

$$(M, \nabla, \text{+ axioms}) \quad \left\{ \quad M, \Gamma(\omega), \Gamma(a\omega) = a\Gamma(\omega)\bar{a}' - d\bar{a}\bar{a}' \right.$$

$$\left. \begin{aligned} \Gamma^{\mu}_{\nu}(\omega) &= \Gamma^{\mu}_{\nu\sigma} \omega^{\sigma} \\ \nabla_{X_{\mu}} \omega^{\nu} &= -\Gamma^{\nu}_{\sigma\mu} \omega^{\sigma} \end{aligned} \right\} \begin{array}{l} \text{in LOCAL FRAME} \\ X_{\mu} \leftrightarrow \omega^{\mu} \end{array}$$

Torsion:

$$T \in \mathcal{X}(M)^1_2$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \quad \left\{ \begin{array}{l} \text{Canonical 1-form of type id} \\ \theta^{\mu}(\omega) = \omega^{\mu} \end{array} \right.$$

$$T(X, Y) = \Theta^{\mu}(X, Y) X_{\mu}$$

Curvature:

$$R \in \mathcal{X}(M)^1_3$$

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z \quad \left\{ \begin{array}{l} \Omega = d\Gamma + \Gamma \wedge \Gamma \end{array} \right.$$

$$R(X, Y)Z = \Omega^{\alpha}_{\beta}(X, Y) \theta^{\beta}(Z) X_{\alpha}$$

Bianchi identity

$$D\Theta^{\mu} = \Omega^{\mu}_{\nu} \wedge \theta^{\nu} \quad \text{I}^{\text{st}}$$

$$D\Omega^{\mu}_{\nu} = 0 \quad \text{II}^{\text{nd}}$$

If $T = 0$ then

$$\text{II}^{\text{nd}} \text{ is } R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$$

$$\text{I}^{\text{st}} \text{ is } (\nabla_X R)(Y, Z) + (\nabla_Z R)(X, Y) + (\nabla_Y R)(Z, X) = 0$$

Check 0-form of type $\binom{1}{3}$

$$\begin{aligned}
 X_\alpha \lrcorner X_\beta \lrcorner \Omega^\mu_\nu(\omega) &= X_\alpha \lrcorner X_\beta \lrcorner (d\Gamma^\mu_\nu(\omega) + \Gamma^\mu_\rho(\omega) \wedge \Gamma^\rho_\nu(\omega)) = \\
 &= X_\alpha \lrcorner X_\beta \lrcorner (d\Gamma^\mu_{r\beta} \omega^\beta + \Gamma^\mu_{r\beta} d\omega^\beta + \dots) = \\
 &= X_\alpha \lrcorner (X_\beta (\Gamma^\mu_{r\beta}) \omega^\beta - d\Gamma^\mu_{r\beta} + \dots) = \\
 &= \underbrace{X_\beta (\Gamma^\mu_{r\alpha}) - X_\alpha (\Gamma^\mu_{r\beta}) + \dots}_{\substack{\downarrow \text{0-form of type } \binom{1}{3}}}
 \end{aligned}$$

can be written as;

$$R(X_\beta, X_\alpha) X_\nu \stackrel{\text{II}}{=} \underbrace{\left(X_\alpha \lrcorner X_\beta \lrcorner \tilde{\Omega}^\mu_\nu(\omega) \right)}_{\substack{\downarrow \text{0-form of type } \binom{1}{3}}} X_\mu$$

$$\begin{aligned}
 \nabla_{X_\beta} \nabla_{X_\alpha} X_\nu - \nabla_{X_\alpha} \nabla_{X_\beta} X_\nu - \nabla_{[X_\beta, X_\alpha]} X_\nu &= \\
 = \nabla_{X_\beta} (\Gamma^s_{r\alpha} X_\nu) - \nabla_{X_\alpha} (\Gamma^s_{r\beta} X_\nu) - C^s_{\rho\alpha} \nabla_{X_\rho} X_\nu &= \\
 = X_\beta (\Gamma^s_{r\alpha}) X_\nu - X_\alpha (\Gamma^s_{r\beta}) X_\nu - C^s_{\rho\alpha} \Gamma^a_{r\beta} X_\nu + \dots &= \\
 = \left(X_\beta (\Gamma^\mu_{r\alpha}) - X_\alpha (\Gamma^\mu_{r\beta}) + \dots \right) X_\mu &
 \end{aligned}$$

$$\Rightarrow \Omega^\mu_\nu = \tilde{\Omega}^\mu_\nu$$



$$\mathbb{H}^{\mu} = \frac{1}{2} Q^{\mu}_{rs} \theta^r \wedge \theta^s$$

$$\Omega^{\mu}_{r\sigma} = \frac{1}{2} R^{\mu}_{r\sigma\gamma} \theta^s \wedge \theta^{\gamma}$$

$$Q^{\mu}_{rs} - 0\text{-form of type } (1, 2)$$

$$R^{\mu}_{r\sigma\gamma} - 0\text{-form of type } (1, 3)$$

In general:

α^A - 0-form of type s :

$$\Rightarrow \boxed{D\alpha^A = \nabla_{\mu} \alpha^A \theta^{\mu}}; \quad \nabla_{\mu} \alpha^A = \nabla_{x^{\mu}} \alpha^A.$$

$$D\mathbb{H}^{\mu} = \frac{1}{2} DQ^{\mu}_{rs} \theta^r \wedge \theta^s + \frac{1}{2} Q^{\mu}_{rs} D\theta^r \wedge \theta^s + \\ - \frac{1}{2} Q^{\mu}_{rs} \theta^r \wedge D\theta^s =$$

$$= \frac{1}{2} \nabla_{\alpha} Q^{\mu}_{rs} \theta^{\alpha} \wedge \theta^r \wedge \theta^s + \frac{1}{2} Q^{\mu}_{rs} \frac{1}{2} Q^{\nu}_{\alpha\beta} \theta^{\alpha} \wedge \theta^{\beta} \wedge \theta^s + \\ - \frac{1}{2} Q^{\mu}_{rs} \theta^r \wedge \frac{1}{2} Q^{\beta}_{\alpha\gamma} \theta^{\alpha} \wedge \theta^{\gamma} =$$

$$= \frac{1}{2} \left[\nabla_{\alpha} Q^{\mu}_{rs} + \frac{1}{2} Q^{\mu}_{rs} Q^{\beta}_{\alpha\gamma} + \frac{1}{2} Q^{\mu}_{r\beta} Q^{\beta}_{\alpha\gamma} \right] \theta^{\alpha} \wedge \theta^r \wedge \theta^s \\ - \frac{1}{2} Q^{\mu}_{pr} Q^{\beta}_{\alpha\gamma}$$

$$\Omega^{\mu}_{r\lambda\theta^r} = \frac{1}{2} R^{\mu}_{r\alpha\beta} \theta^{\alpha} \wedge \theta^{\beta} \wedge \theta^r = -\frac{1}{2} R^{\mu}_{r\alpha\beta} \theta^{\alpha} \wedge \theta^r \wedge \theta^{\beta}$$

$$\boxed{\nabla_{\alpha} Q^{\mu}_{rs} + \frac{1}{2} Q^{\mu}_{\beta\gamma} Q^{\beta}_{\alpha\gamma} - \frac{1}{2} Q^{\mu}_{\beta\gamma} Q^{\beta}_{\alpha\gamma} = \\ = -R^{\mu}_{[r\alpha\beta]}}$$

$$\text{If } \mathbb{H}^{\mu} = 0 \Leftrightarrow \boxed{R^{\mu}_{[r\alpha\beta]} = 0}$$

$$D\Omega^\mu_\nu = \frac{1}{2} D(R^\mu_{\nu\sigma\alpha} \theta^\sigma \wedge \theta^\alpha) = \\ = \nabla_\alpha R^\mu_{\nu\sigma\alpha} \theta^\sigma \wedge \theta^\alpha + \dots$$

If $\Theta^\mu = 0$

$$D\Omega^\mu_\nu = 0 \Leftrightarrow \boxed{R^\mu_{\nu[\sigma;\alpha]} = 0}$$

Then

If $\Theta^\mu = 0$ ($T \equiv 0$) then

$$\begin{cases} R^\mu_{[\nu\alpha\beta]} = 0 & \text{II}^{\text{nd}} \text{ Bianchi} \\ R^\mu_{\nu[\sigma;\alpha]} = 0 & \text{I}^{\text{st}} \text{ Bianchi} \end{cases}$$

Assume that $T=0$ or $\Theta^\mu=0$.

$$\Rightarrow \boxed{\Omega^\mu_\nu \wedge \theta^\nu = 0}$$

Let's check:

$$\begin{aligned} & R(V,Y)Z + R(Z,V)Y + R(Y,Z)V = \\ &= \Omega^\alpha_\beta(V,Y) \theta^\beta(Z) X_\alpha + \Omega^\alpha_\beta(Z,V) \theta^\beta(Y) X_\alpha + \Omega^\alpha_\beta(Y,Z) \theta^\beta(V) X_\alpha = \\ &= (R^\alpha_{\beta\mu\nu} \underline{V^\mu Y^\nu} Z^\beta + R^\alpha_{\beta\mu\nu} Z^\mu V^\nu Y^\beta + R^\alpha_{\beta\mu\nu} Y^\mu Z^\nu V^\beta) X_\alpha = \\ &= [R^\alpha_{\beta\gamma\mu} + R^\alpha_{\nu\beta\mu} + R^\alpha_{\mu\nu\beta}] V^\mu Y^\nu Z^\beta X_\alpha \\ &= 2 R^\alpha_{[\beta\gamma\mu]} V^\mu Y^\nu Z^\beta X_\alpha = 0 \end{aligned}$$

$$\nabla_x R = X^\mu (\nabla_\mu R^\alpha_{\beta\gamma\delta}) X_\alpha \otimes \omega^\beta \otimes \omega^\gamma \otimes \omega^\delta$$

$$\begin{aligned} (\nabla_x R)(Y, Z) &= X^\mu \nabla_\mu R^\alpha_{\beta\gamma\delta} Y^\beta Z^\gamma X_\alpha \otimes \omega^\delta \\ &= X^\mu Y^\beta Z^\gamma \nabla_\mu R^\alpha_{\beta\gamma\delta} X_\alpha \otimes \omega^\delta \end{aligned}$$

$$(\nabla_z R)(X, Y) = X^\mu Y^\beta Z^\gamma \nabla_\delta R^\alpha_{\beta\mu\gamma} X_\alpha \otimes \omega^\delta$$

$$(\nabla_y R)(Z, X) = X^\mu Y^\beta Z^\gamma \nabla_\gamma R^\alpha_{\beta\delta\mu} X_\alpha \otimes \omega^\delta$$

$$\Rightarrow (\nabla_x R)(Y, Z) + (\nabla_z R)(X, Y) + (\nabla_y R)(Z, X) =$$

$$= X^\mu Y^\beta Z^\gamma \underbrace{[\nabla_\mu R^\alpha_{\beta\gamma\delta} + \nabla_\delta R^\alpha_{\beta\mu\gamma} + \nabla_\gamma R^\alpha_{\beta\delta\mu}]}_{\substack{= \\ 0}} X_\alpha \otimes \omega^\delta$$