Manifolds with affine connection ∇

Two languages:

$(M, \nabla, + \text{axioms})$ | $M, \Gamma(\omega), \Gamma(a\omega) = a\Gamma(\omega)a^{-1} - da\,a^{-1})$

$$\left.\begin{aligned} \Gamma^\mu{}_\nu(\omega) &= \Gamma^\mu{}_{\nu\rho}\,\omega^\rho \\ \nabla_{X_\mu}\omega^\nu &= -\Gamma^\nu{}_{\rho\mu}\,\omega^\rho \end{aligned}\right\} \text{in LOCAL FRAME } X_\mu \longleftrightarrow \omega^\mu$$

Torsion:

$T \in \mathfrak{X}(M)^1_2$

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

Cannonical 1-form of type id

$\theta^\mu(\omega) = \omega^\mu$

$\Theta^\mu = D\theta^\mu = d\theta^\mu + \Gamma^\mu{}_\nu \wedge \theta^\nu$

$$T(X,Y) = \Theta^\mu(X,Y)X_\mu$$

Curvature:

$R \in \mathfrak{X}(M)^1_3$:

$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$

$\Omega = d\Gamma + \Gamma \wedge \Gamma$

$$R(X,Y)Z = \Omega^\alpha{}_\beta(X,Y)\theta^\beta(Z)X_\alpha$$

Bianchi identity

$D\Theta^\mu = \Omega^\mu{}_\nu \wedge \theta^\nu$ $\quad$ I st

$D\Omega^\mu{}_\nu = 0$ $\quad$ II nd

If $T = 0$ then

II nd is $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$

I st is $(\nabla_X R)(Y,Z) + (\nabla_Z R)(X,Y) + (\nabla_Y R)(Z,X) = 0$

<u>Check</u>

$$X_\alpha \lrcorner X_\beta \lrcorner \Omega^\mu{}_\nu(\omega) = X_\alpha \lrcorner X_\beta \lrcorner \big(\overset{\text{0-form of type } \binom{1}{3}}{\downarrow}\; d\Gamma^\mu{}_\nu(\omega) + \Gamma^\mu{}_\rho(\omega)\wedge\Gamma^\rho{}_\nu(\omega)\big) =$$

$$= X_\alpha \lrcorner X_\beta \lrcorner \left(d\Gamma^\mu{}_{\nu\rho}\,\omega^\rho + \Gamma^\mu{}_{\nu\rho}\, d\omega^\rho + \dots \right) =$$

$$= X_\alpha \lrcorner \left(X_\beta(\Gamma^\mu{}_{\nu\rho})\,\omega^\rho - d\Gamma^\mu{}_{\nu\beta} + \dots \right) =$$

$$= \underbrace{X_\beta(\Gamma^\mu{}_{\nu\alpha}) - X_\alpha(\Gamma^\mu{}_{\nu\beta}) + \dots}$$

can be written as:

$$R(X_\beta, X_\alpha)X_\nu \overset{?}{=} \Big(\overset{\text{0-form of type } \binom{1}{3}}{\overbrace{X_\alpha \lrcorner X_\beta \lrcorner \tilde{\Omega}^\mu{}_\nu(\omega)}}\Big) X_\mu$$

$$\parallel$$

$$\nabla_{X_\beta}\nabla_{X_\alpha}X_\nu - \nabla_{X_\alpha}\nabla_{X_\beta}X_\nu - \nabla_{[X_\beta, X_\alpha]}X_\nu =$$

$$= \nabla_{X_\beta}\left(\Gamma^\rho{}_{\nu\alpha}X_\rho\right) - \nabla_{X_\alpha}\left(\Gamma^\rho{}_{\nu\beta}X_\rho\right) - C^\rho{}_{\beta\alpha}\nabla_{X_\rho}X_\nu =$$

$$= X_\beta(\Gamma^\rho{}_{\nu\alpha})X_\rho - X_\alpha(\Gamma^\rho{}_{\nu\beta})X_\rho - C^\rho{}_{\beta\alpha}\Gamma^\sigma{}_{\nu\rho}X_\sigma + \dots =$$

$$= \Big(\underbrace{X_\beta(\Gamma^\mu{}_{\nu\alpha}) - X_\alpha(\Gamma^\mu{}_{\nu\beta}) + \dots}\Big) X_\mu$$

$\checkmark$

$$\Rightarrow \quad \Omega^\mu{}_\nu = \tilde{\Omega}^\mu{}_\nu$$

$$\Theta^\mu = \tfrac{1}{2} Q^\mu{}_{\nu\rho}\, \theta^\nu \wedge \theta^\rho$$

$$\Omega^\mu{}_\nu = \tfrac{1}{2} R^\mu{}_{\nu\rho\sigma}\, \theta^\rho \wedge \theta^\sigma$$

$Q^\mu{}_{\nu\rho}$ - 0-form of type $\binom{1}{2}$

$R^\mu{}_{\nu\rho\sigma}$ - 0-form of type $\binom{1}{3}$

In general:

α^A - 0-form of type ρ:

$$\Rightarrow \quad \boxed{D\alpha^A = \nabla_\mu \alpha^A\, \theta^\mu} \; ; \quad \nabla_\mu \alpha^A = \nabla_{X_\mu} \alpha^A .$$

$$D\Theta^\mu = \tfrac{1}{2} DQ^\mu{}_{\nu\rho} \wedge \theta^\nu \wedge \theta^\rho + \tfrac{1}{2} Q^\mu{}_{\nu\rho}\, D\theta^\nu \wedge \theta^\rho +$$

$$- \tfrac{1}{2} Q^\mu{}_{\nu\rho}\, \theta^\nu \wedge D\theta^\rho =$$

$$= \tfrac{1}{2} \nabla_\alpha Q^\mu{}_{\nu\rho}\, \theta^\alpha \wedge \theta^\nu \wedge \theta^\rho + \tfrac{1}{2} Q^\mu{}_{\nu\rho}\, \tfrac{1}{2} Q^\nu{}_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \wedge \theta^\rho +$$

$$- \tfrac{1}{2} Q^\mu{}_{\nu\rho}\, \theta^\nu \wedge \tfrac{1}{2} Q^\rho{}_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta =$$

$$= \tfrac{1}{2} \left[\nabla_\alpha Q^\mu{}_{\nu\rho} + \tfrac{1}{2} Q^\mu{}_{\beta\rho} Q^\beta{}_{\alpha\nu} + \tfrac{1}{2} Q^\mu{}_{\nu\beta} Q^\beta{}_{\alpha\rho} \right] \theta^\alpha \wedge \theta^\nu \wedge \theta^\rho$$

$$- \tfrac{1}{2} Q^\mu{}_{\beta\nu} Q^\beta{}_{\alpha\rho}$$

$$\Omega^\mu{}_\nu \wedge \theta^\nu = \tfrac{1}{2} R^\mu{}_{\nu\alpha\rho}\, \theta^\alpha \wedge \theta^\rho \wedge \theta^\nu = -\tfrac{1}{2} R^\mu{}_{\nu\alpha\rho}\, \theta^\alpha \wedge \theta^\nu \wedge \theta^\rho$$

$$\boxed{\nabla_{[\alpha} Q^\mu{}_{\nu\rho]} + \tfrac{1}{2} Q^\mu{}_{\beta[\rho} Q^\beta{}_{\alpha\nu]} - \tfrac{1}{2} Q^\mu{}_{\beta[\nu} Q^\beta{}_{\alpha\rho]} = -R^\mu{}_{[\nu\alpha\rho]}}$$

If $\Theta^\mu = 0 \iff \boxed{R^\mu{}_{[\nu\alpha\rho]} = 0}$

$$D\Omega^\mu{}_\nu = \frac{1}{2} D\left(R^\mu{}_{\nu\rho\sigma}\,\theta^\rho \wedge \theta^\sigma\right) =$$

$$= \nabla_\alpha R^\mu{}_{\nu\rho\sigma}\,\theta^\alpha \wedge \theta^\rho \wedge \theta^\sigma + \dots$$

If $\Theta^\mu = 0$

$$D\Omega^\mu{}_\nu = 0 \iff \boxed{R^\mu{}_{\nu[\rho\sigma;\alpha]} = 0}$$

<u>Thm</u>

If $\Theta^\mu = 0$ $(T \equiv 0)$ then

$$\begin{cases} R^\mu{}_{[\nu\alpha\rho]} = 0 & \text{II}^{nd} \text{ Bianchi} \\ R^\mu{}_{\nu[\rho\sigma;\alpha]} = 0 & \text{I}^{st} \text{ Bianchi} \end{cases}$$

Assume that $T = 0$ or $\Theta^\mu = 0$.

$$\Rightarrow \boxed{\Omega^\mu{}_\nu \wedge \theta^\nu = 0}$$

Let's check:

$$R(V,Y)Z + R(Z,V)Y + R(Y,Z)V =$$

$$= \Omega^\alpha{}_\beta(V,Y)\,\theta^\beta(Z)X_\alpha + \Omega^\alpha{}_\beta(Z,V)\,\theta^\beta(Y)X_\alpha + \Omega^\alpha{}_\beta(Y,Z)\,\theta^\beta(V)X_\alpha =$$

$$= \left(R^\alpha{}_{\beta\mu\nu}\,\underline{V^\mu Y^\nu Z^\beta} + R^\alpha{}_{\beta\mu\nu}\,Z^\mu V^\nu Y^\beta + R^\alpha{}_{\beta\mu\nu}\,Y^\mu Z^\nu V^\beta\right) X_\alpha =$$

$$= \left[R^\alpha{}_{\beta\nu\mu} + R^\alpha{}_{\nu\beta\mu} + R^\alpha{}_{\mu\nu\beta}\right] V^\mu Y^\nu Z^\beta X_\alpha$$

$$= 2\, R^\alpha{}_{[\beta\nu\mu]}\, V^\mu Y^\nu Z^\beta X_\alpha = 0$$

$$\nabla_X R = X^\mu \left(\nabla_\mu R^\alpha{}_{\beta\gamma\delta}\right) X_\alpha \otimes \omega^\beta \otimes \omega^\gamma \otimes \omega^\delta$$

$$(\nabla_X R)(Y,Z) = X^\mu \, \nabla_\mu R^\alpha{}_{\beta\gamma\delta} Y^\gamma Z^\delta \; X_\alpha \otimes \omega^\beta =$$

$$= X^\mu Y^\gamma Z^\delta \, \nabla_\mu R^\alpha{}_{\beta\gamma\delta} \; X_\alpha \otimes \omega^\beta$$

$$(\nabla_Z R)(X,Y) = X^\mu Y^\gamma Z^\delta \, \nabla_\delta R^\alpha{}_{\beta\mu\gamma} \; X_\alpha \otimes \omega^\beta$$

$$(\nabla_Y R)(Z,X) = X^\mu Y^\gamma Z^\delta \, \nabla_\gamma R^\alpha{}_{\beta\delta\mu} \; X_\alpha \otimes \omega^\beta$$

$$\Rightarrow \quad (\nabla_X R)(Y,Z) + (\nabla_Z R)(X,Y) + (\nabla_Y R)(Z,X) =$$

$$= X^\mu Y^\gamma Z^\delta \Big[\underbrace{\nabla_\mu R^\alpha{}_{\beta\gamma\delta} + \nabla_\delta R^\alpha{}_{\beta\mu\gamma} + \nabla_\gamma R^\alpha{}_{\beta\delta\mu}}_{=\,0} \Big] X_\alpha \otimes \omega^\beta$$